Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

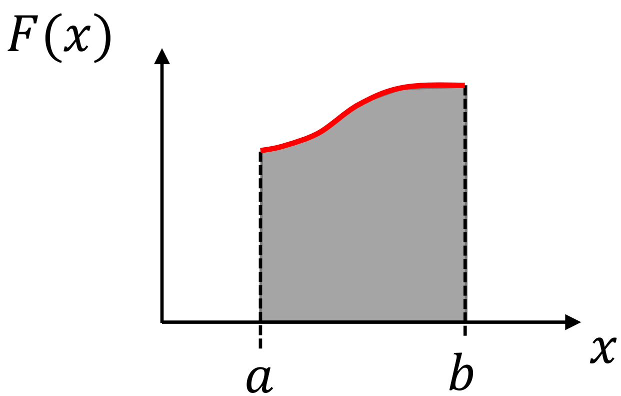
Lecture 5A. Numerical integration

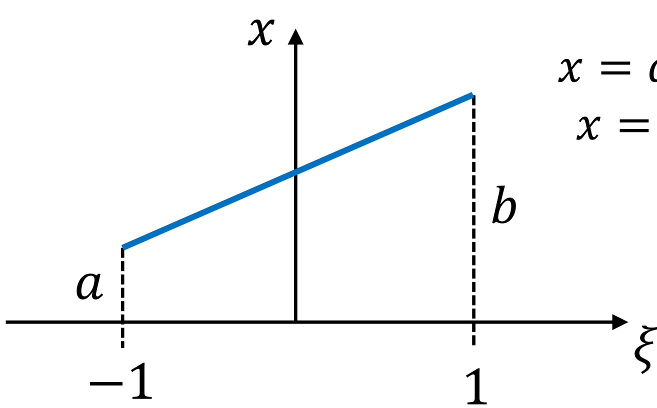
03.2025

Defined integral

x – cartesian coordinate

ξ – natural coordinate





$$x = a \longrightarrow \xi = -1$$
$$x = b \longrightarrow \xi = 1$$

normalization of the function $F(x)$:

$$x(\xi) = \frac{b-a}{2}\,\xi + \frac{a+b}{2} \quad ; \quad f(\xi) = F\left(\frac{b-a}{2}\,\xi + \frac{a+b}{2}\right) \quad ; \quad dx = \frac{b-a}{2}\,d\xi$$

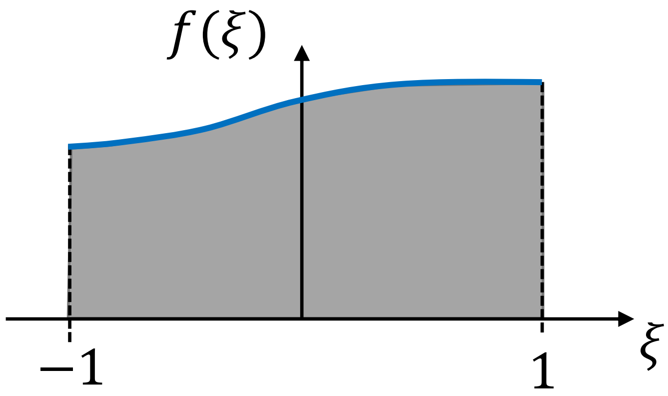
definite integral of the function $F(x)$:

$$\int_a^b F(x)dx = \int_{-1}^{1} f(\xi)\frac{b-a}{2}\,d\xi = \frac{b-a}{2}\int_{-1}^{1} f(\xi)d\xi$$

Gaussian quadrature rule

quadrature rule:

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{n} w_i \cdot f(\xi_i) + R_n$$



n − no. of sample points,
ξ_i −coordinates of sample points
w_i −weight coefficients
R_n −rest of the sum

$$R_n = 0 \quad \Rightarrow \quad \frac{d^{2n}f}{d\xi^{2n}} = 0$$

Numerical integration gives
the exact value of the integral
for polynomials up to $(\mathbf{2n-1})$ degree.

Gaussian quadrature rule for polynomial functions

$R_n = 0 \quad \Rightarrow \quad \frac{d^{2n} f}{d\xi^{2n}} = 0$

for a linear function: $f(\xi) = \alpha\,\xi + \beta \quad ; \quad \frac{df}{d\xi} = \alpha \quad ; \quad \frac{d^2 f}{d\xi^2} = 0 \longrightarrow$ $2n = 2 \longrightarrow \boldsymbol{n = 1}$

One point is enough!

$$\int_{-1}^{1} (\alpha\,\xi + \beta) d\xi = w_1 \cdot f(\xi_1) + 0$$

for one Gaussian point: $\xi_1 = 0\,, w_1 = 2$ ➡ $\int_{-1}^{1} (\alpha\,\xi + \beta) d\xi = w_1 \cdot f(0)$

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polynomial functions:

2nd order: $f(\xi) = \alpha\,\xi^2 + \beta\,\xi + \gamma \; ; \; \frac{df}{d\xi} = 2\alpha\,\xi + \beta \; ; \; \frac{d^2 f}{d\xi^2} = 2\alpha \; ; \; \frac{d^3 f}{d\xi^3} = 0 \longrightarrow$

$2n = 3 \longrightarrow n = 1.5 \longrightarrow \boldsymbol{n = 2}$

Two points needed!

for 2 Gauss points: $\xi_1 = -\frac{1}{\sqrt{3}} \; ; \quad \xi_2 = \frac{1}{\sqrt{3}} \quad ; \quad w_1 = w_2 = 1$

$$\int_{-1}^{1} (\alpha\xi^2 + \beta\,\xi + \gamma) d\xi = w_1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + w_2 \cdot f\left(\frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Gaussian quadrature rule for polynomial functions

3th order: $f(\xi) = \alpha\,\xi^3 + \beta\,\xi^2 + \gamma\,\xi + \delta$; $\frac{d^4 f}{d\xi^4} = 0$ $\longrightarrow$ $\boldsymbol{n = 2}$ (2 points)

Two points are enough!

$$\int_{-1}^{1} (\alpha\,\xi^3 + \beta\,\xi^2 + \gamma\,\xi + \delta)\,d\xi = w_1 \cdot f\left(-\frac{1}{\sqrt{3}}\right) + w_2 \cdot f\left(\frac{1}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

4th order: $f(\xi) = \alpha\,\xi^4 + \beta\,\xi^3 + \gamma\,\xi^2 + \delta\,\xi + \varphi$; $\frac{d^5 f}{d\xi^5} = 0$

$$2n = 5 \longrightarrow n = 2.5 \longrightarrow \boldsymbol{n = 3}$$

Three points needed!

for three Gauss points: $\xi_1 = -\sqrt{0.6}$; $\xi_2 = 0$; $\xi_3 = \sqrt{0.6}$;

$w_1 = w_3 = \frac{5}{9}$; $w_2 = \frac{8}{9}$

$$\int_{-1}^{1} (\alpha\,\xi^4 + \beta\,\xi^3 + \gamma\,\xi^2 + \delta\,\xi + \varphi)\,d\xi =$$

$$= \frac{5}{9} \cdot f\left(-\sqrt{0.6}\right) + \frac{8}{9} \cdot f(0) + \frac{5}{9} \cdot f\left(\sqrt{0.6}\right)$$

Gaussian quadrature rule for polynomial functions

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{i=1}^{n} w_i \cdot f(\xi_i) + R_n$$

$$R_n = 0 \quad \Rightarrow \quad \frac{d^{2n}f}{d\xi^{2n}} = 0$$

Polynomial degree	Number of Gauss points	ξ_i	w_i
1	1	0	2
3	2	$-1/\sqrt{3}$ $+1/\sqrt{3}$	1 1
5	3	$-\sqrt{0.6}$ 0 $+\sqrt{0.6}$	5/9 8/9 5/9
7	4	-0.861136311594953 -0.339981043584856 $+0.339981043584856$ $+0.861136311594953$	0.347854845137454 0.652145154862546 0.652145154862546 0.347854845137454

The sum of the weighting factors is always 2.
Numerical integration gives the exact value of the integral for polynomials up to $(\mathbf{2n-1})$ degree

Gaussian quadrature rule for 2D FEs

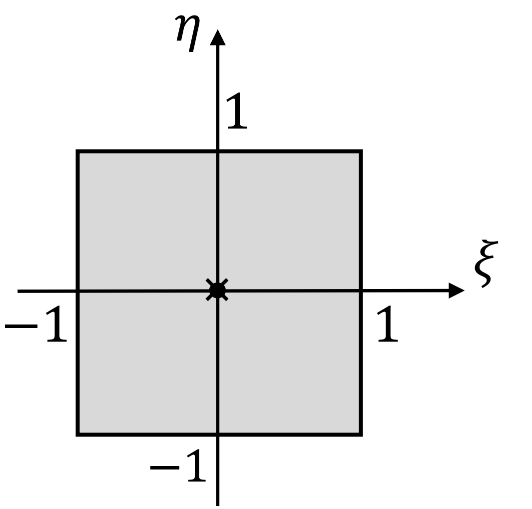
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi d\eta = \int_{-1}^{1}\left(\sum_{i=1}^{n}\left(w_i \cdot f(\xi_i,\eta)\right)\right)d\eta =$$

$$= \sum_{j=1}^{n} w_j \sum_{i=1}^{n}\left(w_i \cdot f(\xi_i,\eta_j)\right) = \sum_{j=1}^{n}\sum_{i=1}^{n}\left(w_i w_j \cdot f(\xi_i,\eta_j)\right)$$

For one Gaussian point we have:

$\boldsymbol{n = 1}$: $\xi_1 = \eta_1 = 0\,,\quad w_1 = 2$

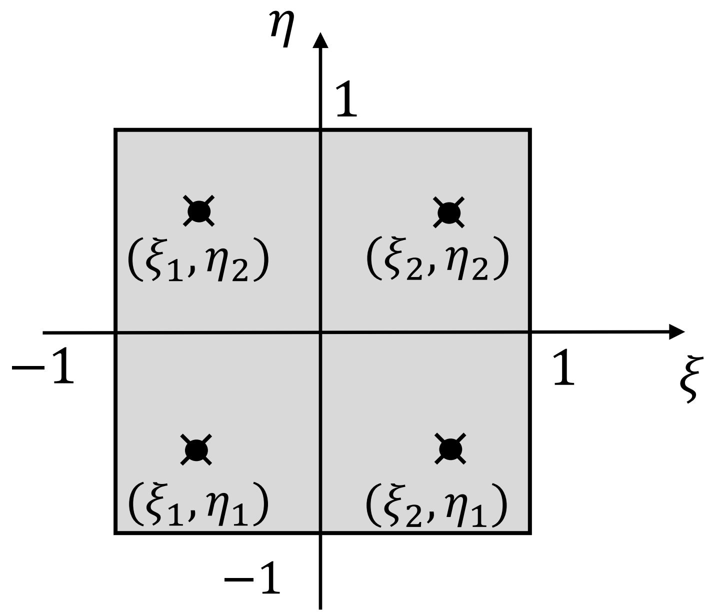
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi d\eta = w_1 w_1 \cdot f(0,0) = 4f(0,0)$$



Gaussian quadrature rule for 2D FEs

For two Gauss points on each direction we have:

$$n = 2: \quad \xi_1 = \eta_1 = -\frac{1}{\sqrt{3}}, \quad \xi_2 = \eta_2 = \frac{1}{\sqrt{3}} \ ; \ w_1 = w_2 = 1$$



$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta =$$

$$= w_1 w_1 \cdot f(\xi_1,\eta_1) + w_2 w_1 \cdot f(\xi_2,\eta_1) + w_2 w_2 \cdot f(\xi_2,\eta_2) + w_1 w_2 \cdot f(\xi_1,\eta_2) =$$

$$= f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

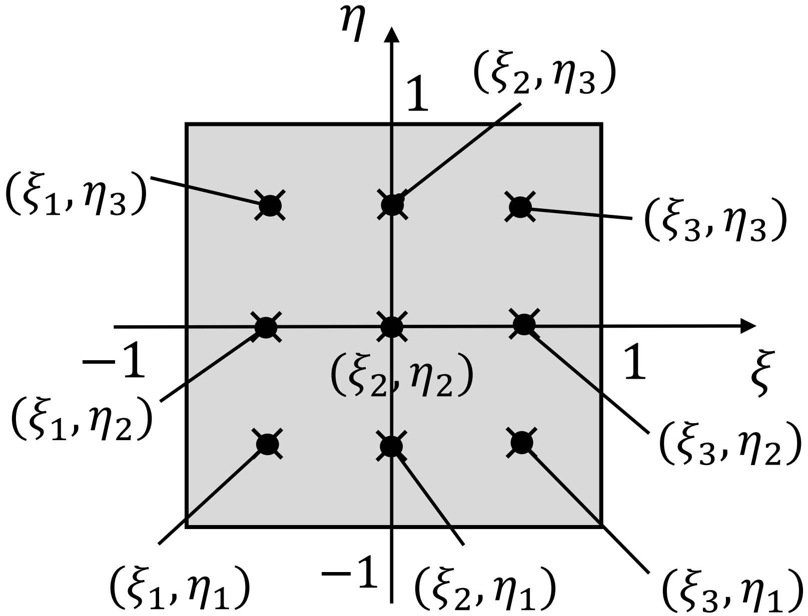
Gaussian quadrature rule for 2D FEs

For three Gauss points on each direction we have:

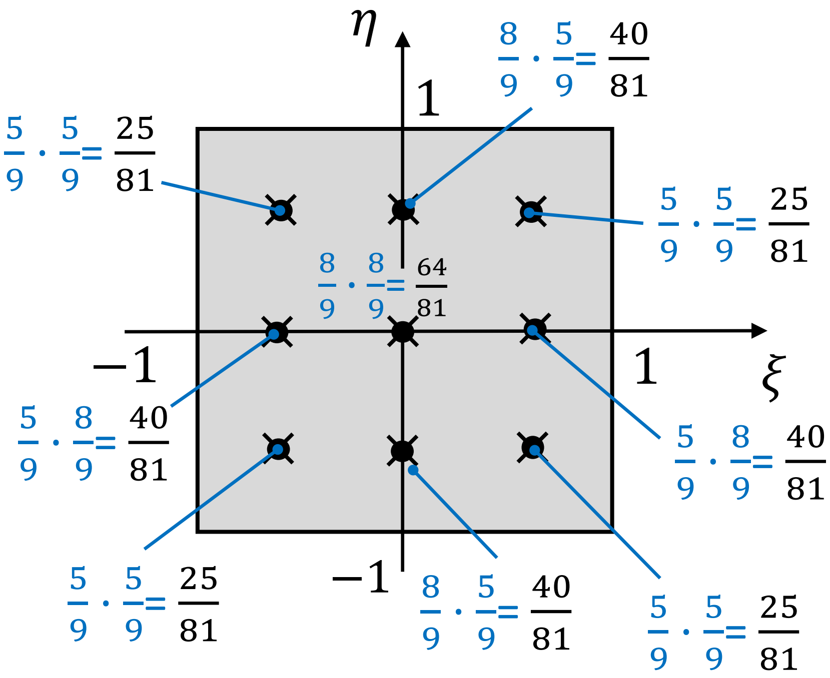
$\boldsymbol{n = 3}$:

$$\xi_1 = \eta_1 = -\sqrt{0.6}\ , \quad \xi_2 = \eta_2 = 0 \quad , \quad \xi_3 = \eta_3 = \sqrt{0.6}$$

$$w_1 = w_3 = \frac{5}{9} \ ; \ w_2 = \frac{8}{9}$$



(ξ_i, η_j)



$w_i w_j$

Gaussian quadrature rule for 2D FEs

For three Gauss points on each direction we have:

$$\boldsymbol{n = 3}: \quad \xi_1 = \eta_1 = -\sqrt{0.6}\ , \quad \xi_2 = \eta_2 = 0 \quad , \quad \xi_3 = \eta_3 = \sqrt{0.6}$$

$$w_1 = w_3 = \frac{5}{9} \ ; \ w_2 = \frac{8}{9}$$

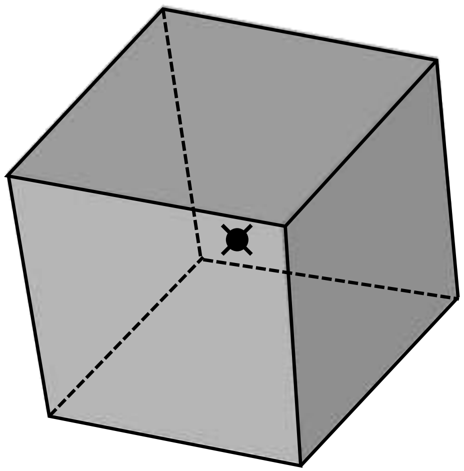
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta =$$

$$= w_1 w_1 \cdot f(\xi_1,\eta_1) + w_2 w_1 \cdot f(\xi_2,\eta_1) + w_3 w_1 \cdot f(\xi_3,\eta_1) +$$
$$+ w_1 w_2 \cdot f(\xi_1,\eta_2) + w_2 w_2 \cdot f(\xi_2,\eta_2) + w_3 w_2 \cdot f(\xi_3,\eta_2) +$$
$$+ w_1 w_3 \cdot f(\xi_1,\eta_3) + w_2 w_3 \cdot f(\xi_2,\eta_3) + w_3 w_3 \cdot f(\xi_3,\eta_3) =$$

$$= \frac{5}{9}\cdot\frac{5}{9} f(-\sqrt{0.6}, -\sqrt{0.6}) + \frac{8}{9}\cdot\frac{5}{9} f(0, -\sqrt{0.6}) + \frac{5}{9}\cdot\frac{5}{9} f\left(\sqrt{0.6}, -\sqrt{0.6}\right) +$$
$$+ \frac{5}{9}\cdot\frac{8}{9} f(-\sqrt{0.6}, 0) + \frac{8}{9}\cdot\frac{8}{9} f(0,0) + \frac{5}{9}\cdot\frac{8}{9} f(\sqrt{0.6}, 0) +$$
$$+ \frac{5}{9}\cdot\frac{5}{9} f(-\sqrt{0.6}, \sqrt{0.6}) + \frac{8}{9}\cdot\frac{5}{9} f(0, \sqrt{0.6}) + \frac{5}{9}\cdot\frac{5}{9} f\left(\sqrt{0.6}, \sqrt{0.6}\right)$$

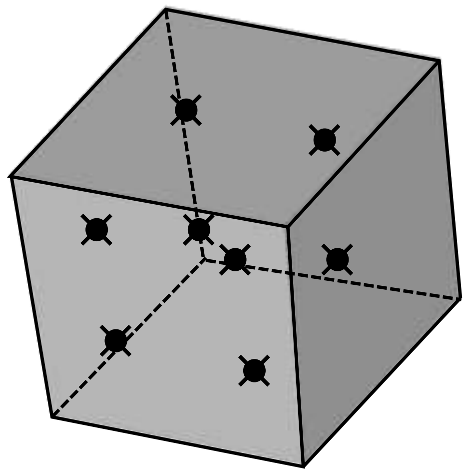
Gaussian quadrature rule for 3D FEs

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta\,)\,d\xi d\eta d\zeta = \sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(w_i w_j w_k \cdot f(\,\xi_i,\eta_i,\zeta_k)\right)$$



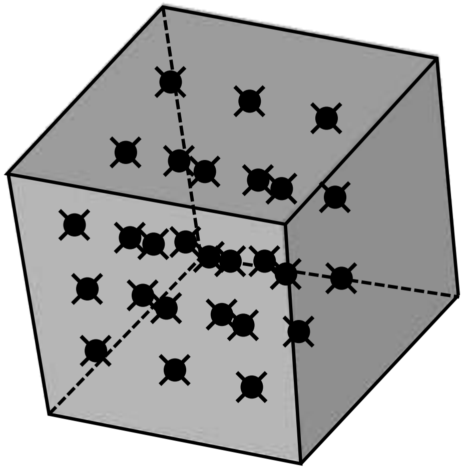
$n = 1$

For one Gaussian point



$n = 2\ \ (2 \times 2 \times 2)$

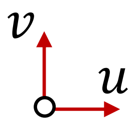
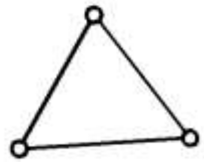
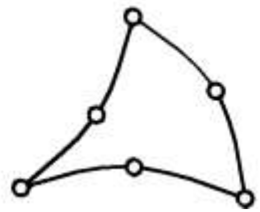
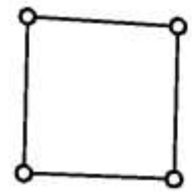
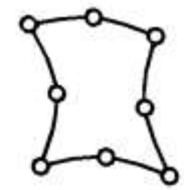
For two Gauss points on each direction



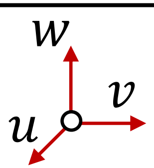
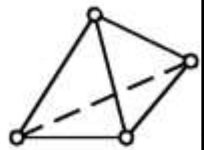
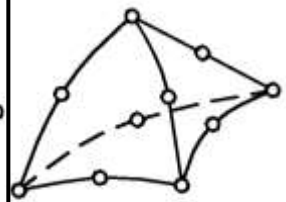
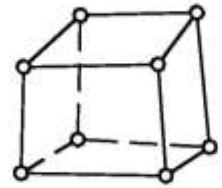
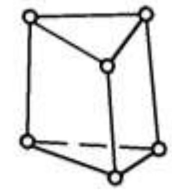
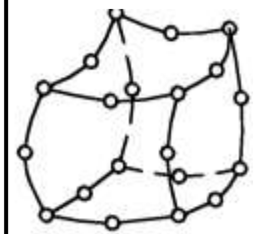
$n = 3\ \ (3 \times 3 \times 3)$

For three Gauss points in each direction

Integration scheme for 2D elements

 2D	3–node	6–node	4–node	8–node
Integration type				
FULL	3	3	2×2	3×3
REDUCED	1	1	1	2×2

Integration scheme for 3D elements

 3D	4–node	10–node	8–node	6–node	20–node
Integration type					
FULL	4	11	$2 \times 2 \times 2$	3×3	$3 \times 3 \times 3$
REDUCED	1	5	1	3×2	$2 \times 2 \times 2$